Universidade de Lisboa - ISEG

Mathematical Economics

FIRST EXAM

January 17, 2023

Maximum duration: 2h30m

PART I

(1) Consider the following correspondence $F : [0,1] \rightrightarrows [0,1]$,

$$F(x) = \begin{cases} \{4x(1-x)\}, & x < \frac{1}{2} \\ [a,b], & x = \frac{1}{2} \\ \{2x-1\}, & x > \frac{1}{2} \end{cases}$$

where $0 \leq a \leq b \leq 1$.

(a) Determine the value of a and b such that F satisfies the assumptions of the Kakutani fixed point theorem. (1 point)

(b) Find the fixed points of F for the value of a and b found in (a). In case you did not solve (a), you may take $a = b = \frac{1}{2}$. (1 point)

(2) Consider the following sets

$$A = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\},$$
$$B = \{(x,y) \in \mathbb{R}^2 \colon |y-3| \leq 1\}.$$

(a) Show that A and B can be separated by a hyperplane. (1 point)

(b) Using the Brouwer fixed point theorem, show that $f \colon A \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{y^2}{2}, \frac{x+y}{2}\right)$$

has a fixed point in A. (1 point)

(c) Compute the fixed point(s) of f. (1 point)

PART II

(1) Find and classify the critical points of

$$f(x,y,z) = x \log y + 2x + z^2 - z, \quad x, z \in \mathbb{R},\ y > 0.$$

(2 points)

(2) Consider the following utility maximization problem. Suppose that there are two goods and an agent with income $m > 1$ who wishes to maximize her utility

$$U(x,y) = \sqrt{xy}, \quad x > 0, \quad y > 0,$$

subject to a budget constraint $px + y \leq m$ where $p \in [0, 1[$.

(a) Write down the necessary conditions to solve the utility maximization problem. (1 point)

(b) Find the solution of the problem assuming $m = 4$ and $p = \frac{1}{2}$. (2 points)

PART III

(1) Consider the initial value problem (IVP)

$$x'(t) - tx(t) = t, \quad x(1) = 0$$

(a) Classify the ODE. (0.5 points)
(b) Solve the IVP. (1.5 points)

(2) Consider the scalar ODE

$$x' = x(1-x)(1+x).$$

(a) Determine and classify the equilibrium points. (0.5 points)
(b) Sketch the phase portrait. (1 point)

(3) Solve the IVP,

$$x'' + 2x' + x = 1, \quad x(0) = -1, \quad x'(0) = 1.$$

(1.5 points)

PART IV

(1) Consider the calculus of variations problem

$$\min_{x(t)} \int_0^1 2e^t x(t) + \dot{x}^2(t)\, dt$$

where $x(0) = 0$ and $x(1)$ is free.

(a) Write the corresponding Euler-Lagrange equation. (1 point)

(b) Find the solution to the calculus of variations problem. (1.5 points)

(2) The one year investment problem for a firm is

$$\max_{I(t)} \int_0^1 \left(K(t) - \frac{I(t)^2}{2} \right) e^{-t} dt,$$

subject to capital accumulation equation, $\dot{K} = I$, the initial condition for the capital stock, $K(0) = K_0 > 0$, and a final threshold $K(1) \geq K_0$. In the model, the variable I represents investment.

(a) Write the first-order optimality conditions according to the Pontryiagin's maximum principle. (0.5 points)

(b) Find the explicit solution to the problem. (2 points)